

Parker's Model for Stellar Wind and Magnetohydrodynamic Extensions

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Abstract

In this paper, we first revisit Parker's [1] hydrodynamic model for a stellar wind and make further analytic considerations. We show that the visualization of an effective de Laval type nozzle associated with Parker's model is valid barring discrepancies with regard to density variations. We then make an analytic considerations on the Weber-Davis [8] magnetohydrodynamic (MHD) extension of Parker's model with a view to provide a qualitative understanding of the coupling between the magnetic field and the plasma motion in the stellar wind. We find that,

- * the MHD azimuthal velocity profile actually resembles that for hydrodynamic Burgers vortex;
- * Keplerian-orbit conditions prevail near a strong rotator even in a magnetized situation;
- * Parker's hydrodynamic scenario [1] seems to reappear in the strong magnetization regime.

1. Introduction

Stellar wind is an interplanetary continuous outflow of hot plasma from a star and an associated remnant of the stellar magnetic field that pervades the space surrounding the star. The high speeds observed in stellar winds are not achievable via control of conditions at the coronal base while thermal conduction is not capable of carrying sufficient energy into the corona to accelerate the wind speeds to $O(100\text{km/sec})$ or greater. It is therefore believed that some additional acceleration mechanism must be operational beyond the coronal base, the understanding of which is one of the most challenging issues of stellar physics research. Parker [1] gave an ingenious stationary model for the stellar wind, which provided a physically realistic solution for the stellar wind which starts with subsonic speeds at the coronal base and expands continuously to supersonic speeds at large distances. In the case of the Sun, the existence of the solar wind was confirmed and its properties were revealed by *in situ* observations from satellites (Hundhausen [2]).

Stellar winds carry off a very negligible amount of mass from the stars while they provide (especially when magnetized) a very effective vehicle to provide angular momentum loss¹ from the stars. The interaction of a protostar's magnetic field with the stellar wind leading to an enhancement of the outward transfer of the angular momentum of the protostar provides a mechanism for the braking of the protostar's rotation rate - *magnetic braking* (Schatzmann [3]). The corona and the stellar wind are fully ionized so the magnetic field lines which are anchored into the coronal base remain *frozen* in the stellar wind as it streams outward, and enforce a *co-rotation*² of the stellar wind as if it were a solid body out to the *Alfvén radius* r_A (where $v_r = B_r/\sqrt{\rho}$)³ with a larger effective moment of inertia. This results in an enhanced outward transfer of angular momentum when the stellar wind eventually escapes at great distances from the star hence spinning the star down⁴.

Near the surface of the star, the magnetization effects are small and the stellar wind follows Parker's hydrodynamic model [1] reasonably well. However, farther away, the magnetization effects become strong⁵ The interaction of the stellar magnetic field with the expanding coronal plasma is therefore a major topic in stellar wind research. The magnetohydrodynamic (MHD) version of Parker's hydrodynamic model was considered by Weber and Davis [8] and Belcher and MacGregor [9] to treat the magnetized stellar winds. In the interest of analytic simplicity, Weber and Davis [8] considered a model with complete axial symmetry in which the stellar wind distributes the magnetic field in the equatorial plane of the star with no

¹The angular momentum loss is very crucial for a protostar because the protostar would otherwise breakup due to huge centrifugal forces at the equator that develop during the condensation process of the parental gas cloud.

²In the region $r \lesssim r_A$ the strong magnetic field forces the wind co-rotate with the star. However, in the region $r \gtrsim r_A$, with a weak magnetic field there is negligible magnetic torque to spin up the stellar wind; the angular momentum conservation in the wind then causes the azimuthal speed to decrease outward like $1/r$; as a result, the rotational effects become very weak farther away from the star. The weaker magnetic field frozen in the wind is swept by the wind and is forced to wrap up in a *spiral* pattern in the equatorial plane (Parker [1]). *In situ* observations by *Mariner II* confirmed that the average interplanetary magnetic field (IMF) showed this *spiral* structure (Ness and Wilcox [4]).

³Numerical simulation (ud-Doula et al. [5]) with a dipole magnetic field showed that the Alfvén radius r_A is just a little beyond the maximum extent of closed magnetic loops in the stellar magnetosphere.

⁴This process indeed caused the Sun to lose most of its initial angular momentum while the planets did not lose any.

⁵The magnetic pressure at 1AU is comparable to the gas pressure in the solar wind, so the magnetization effects become as important as the thermal effects in the wind. Indeed, the existence of the solar wind was surmized in the 1950's as a result of the evidence that small variations in the earth's magnetic field (*geomagnetic* activity) were caused by observable phenomena on the Sun (*solar* activity) (example: *aurora borealis* (Birkeland [6]), cometary tails (Biermann [7])).

component normal to this plane⁶. The idealization of a radially expanding stellar wind in a dipole magnetic field by a monopole magnetic field⁷ with spherical symmetry has some merit in illustrating the general principles and giving a rough idea of the overall IMF and flow configuration to be expected. Therefore, the Weber-Davis model [8] can provide a viable physical picture and serve as a good qualitative guide to the real full three-dimensional (3D) wind problem which is prohibitively complex⁸.

In this paper, we first revisit Parker's hydrodynamic model [1] and make further analytic considerations. We show that the visualization of an effective de Laval type nozzle associated with Parker's model is valid barring discrepancies with regard to density variations. We then make analytic considerations on the Weber-Davis MHD model [8] with a view to provide a qualitative understanding of the coupling between the magnetic field and the plasma motion in the stellar wind.

2. Parker's Hydrodynamic Model

Parker's hydrodynamic model [1] provided an ingenious development to show that a gas flow in a perpetual divergent channel (like the interplanetary space), thanks to the presence of a retarding body force (like gravity), can evolve from subsonic (at the coronal base) to supersonic (in the interplanetary space) speeds without requiring a physical throat section (as in a typical rocket nozzle situation). In this model, the stellar wind is represented by a steady, spherically symmetric flow so the flow variables are independent of time and depend only on r , the distance from the star. The flow velocity is taken to be in the radial direction - either inward (accretion model) or outward (wind model). The equation of conservation of mass is

$$\frac{2}{r} + \frac{1}{\rho} \frac{d\rho}{dr} + \frac{1}{v_r} \frac{dv_r}{dr} = 0 \quad (1)$$

standard notation being used.

Assuming the gravitational field to be produced by a central mass M , Euler's equation of momentum balance gives

$$\rho v_r \frac{dv_r}{dr} = -\frac{dp}{dr} - \frac{GM}{r^2} \rho \quad (2)$$

M being the mass of the star and G being the gravitational constant.

The gas is assumed to be a perfect gas obeying an equation of state (which replaces the energy balance equation),

$$p = \rho RT \quad (3)$$

⁶The IMF (especially at IAU) lies on average in the ecliptic plane. However, the meridional flow (which is neglected in the Weber-Davis [8] model) is inevitable due to the rise in the magnetic pressure at low latitudes which causes a diverging flow away from the equatorial plane (Suess and Nerney [10]). Further, since the simplest possible magnetic field configuration is a dipole field, a spherically symmetric MHD model describing the entire corona would look unrealistic.

⁷For the case of the Sun, this assumption is, in fact, somewhat vindicated by the solar wind outflow pulling the Sun's complex multiple magnetic field into an open radial configuration which, a few solar radii from the Sun's surface, roughly resembles a split monopole with opposite polarity above and below the heliopheric current sheet (ud-Doula [5]).

⁸Some improvement on the Weber-Davis model [8] was made by Sakurai [11] by extending the coverage to regions out of the equatorial plane.

R being the perfect gas constant.

For analytic simplicity, the flow is assumed to occur under isothermal⁹ ($T = \text{constant}$) conditions, so (3) becomes

$$p = a^2 \rho \quad (4)$$

where a is the constant speed of sound. The high electron thermal conductivity validates this approximation for a solar-type corona.

We assume next that the flow variables as well as their derivatives vary continuously so there are no shocks anywhere in the region under consideration.

Equations (1), (2), and (4) lead to

$$\left(v_r - \frac{a^2}{v_r}\right) \frac{dv_r}{dr} = \frac{2a^2}{r} - \frac{GM}{r^2} \quad (5a)$$

or

$$\frac{1}{v_r} \frac{dv_r}{dr} = \frac{2a^2(r - r_*)}{r^2(v_r^2 - a^2)} \quad (5b)$$

where,

$$r_* \equiv \frac{GM}{2a^2}.$$

Equation (5) exhibits a physically acceptable smooth solution, for which

$$r = r_* : v_r = a \quad (6)$$

so the numerator and denominator in equation (5b) vanish simultaneously¹⁰. This also implies that the gas flow in a purely divergent channel chokes at $r = r_*$.

The solution of equation (5), upon imposing the smoothness condition (6) is

$$\left(\frac{v_r}{a}\right)^2 - \log\left(\frac{v_r}{a}\right)^2 = 4 \log\left(\frac{r}{r_*}\right) + 4\left(\frac{r_*}{r}\right) - 3 \quad (7a)$$

or

⁹In reality, the solar wind cools while expanding through the interplanetary space.

¹⁰In fact, if one puts

$$y \equiv v_r - a, x \equiv r - r_*$$

the local behavior of the solution near $r = r_*$ is given by

$$yy' \approx \left(\frac{a^2}{r_*^2}\right)x$$

from which,

$$y \approx \left(\frac{a}{r_*}\right)x$$

or

$$\left(\frac{v_r}{a} - 1\right) \approx \left(\frac{r}{r_*} - 1\right)$$

describing smooth acceleration of the flow through the transonic regime.

$$v_r e^{-(v_r^2/2a^2)} = a \left(\frac{r_*}{r} \right)^2 e^{(3/2 - 2r_*/r)}. \quad (7b)$$

(7) exhibits two branches, for $r > r_*$,

- * subsonic (Bondi's [12] accretion problem);
- * supersonic (Parker's [1] wind problem).

For $r \ll r_*$, (7b) leads to

$$r \ll r_* : v_r \approx a \left(\frac{r_*}{r} \right)^2 e^{(3/2 - 2r_*/r)} \quad (8)$$

The corresponding density profile, on using equation (1) reexpressed as

$$\rho r^2 v_r = \rho_0 r_0^2 v_0 \quad (9)$$

with $r = r_0$ at the coronal base, is then

$$\rho = \rho_0 e^{-\frac{2r_*}{r_0} \left(1 - \frac{r_0}{r} \right)} \quad (10)$$

which is simply the one given earlier by Chapman [13] using a hydrostatic model.¹¹ The neglect of the radial flow in the subcritical region $r \ll r_*$ is validated by the dominance of gravity there. The decrease of ρ , as per (10), faster than $1/r^2$, implies increase of v_r with r to comply with equation (9).

For $r \gg r_*$, (7a) leads to

$$r \gg r_* : v_r \approx 2a[\log(r/r_*)]^{1/2} \quad (11)$$

which implies that the radial outflow velocity increases slowly but indefinitely. This unphysical result may be traced to the assumption of isothermality of the wind (which is contingent on the continuous addition of energy and cannot of course happen in reality).

3. Effective de Laval nozzle in the Parker Model?

The continuous acceleration of the stellar wind flow, as per the Parker model [1], from subsonic speeds at the coronal base to supersonic speeds in the interplanetary space immediately leads one to suspect a de Laval type nozzle mechanism (Clauser [15])¹² implicit in the Parker model [1]. Indeed, on a superficial level, one may visualize an effective de Laval type nozzle associated with the Parker model [1] and determine its geometry.

For a one-dimensional gas flow in a de Laval nozzle with cross section area $A = A(x)$, we have (Shivamoggi [16]),

$$\frac{1}{v} \left(1 - \frac{v^2}{a^2} \right) \frac{dv}{dx} = -\frac{1}{A} \frac{dA}{dx}. \quad (12)$$

¹¹Lamers and Cassinelli [14] numerically demonstrated that the corona is almost in hydrostatic equilibrium until close to the critical point (6).

¹²Notwithstanding the conceptual facilitation rendered by the de Laval nozzle analogy in this context, it should be pointed out that this analogy was never intended to provide a framework for serious calculation (Parker, private communication, 2016).

If $\tilde{A} = \tilde{A}(r)$ is the cross section area of an effective de Laval nozzle associated with the Parker model [1], we have from equation (5a),

$$\frac{1}{v_r} \left(1 - \frac{v_r^2}{a^2} \right) \frac{dv_r}{dr} = -\frac{2}{r^2} (r - r_*) = -\frac{1}{\tilde{A}} \frac{d\tilde{A}}{dr} \quad (13)$$

from which we obtain,

$$\tilde{A}(r) = 4\pi r^2 \left[e^{-\frac{2r_*}{r_0} \left(1 - \frac{r_0}{r} \right)} \right]. \quad (14)$$

Comparison of (14) with (10) shows that the multiplicative correction to the divergent channel needed to yield the effective de Laval nozzle associated with the Parker model [1] is just the Chapman [13] hydrostatic density profile, which encapsulates the effects of gravity.

It is also easy to verify the the effective de Laval nozzle cross section area $\tilde{A}(r)$ shows a physical throat section at $r = r_*$, as to be expected. In fact,

$$r \leq r_* : \tilde{A}'(r) = 8\pi (r - r_*) e^{-\frac{2r_*}{r_0} \left(1 - \frac{r_0}{r} \right)} \leq 0 \quad (15)$$

and further,

$$r \approx r_* : \tilde{A}(r) \approx 4\pi e^{2 \left(1 - \frac{r_*}{r_0} \right)} \left[r_*^2 + (\Delta r)^2 \right], \Delta r \equiv r - r_*. \quad (16)$$

So, $\tilde{A}(r)$ has a minimum at $r = r_*$, like a de Laval nozzle!

However, the effective nozzle associated with the Parker model [1] turns out to be largely de Laval type, except for discrepancies with regard to density variations. This may be seen by noting that we have for a one-dimensional gas flow in a de Laval nozzle (Shivamoggi [16]),

$$\frac{1}{\rho} \frac{d\rho}{dx} = -\frac{v^2}{a^2} \frac{1}{v} \left(\frac{dv}{dx} \right) \quad (17)$$

which implies,

$$* v \ll a : \frac{d\rho}{dx} \approx 0 - \text{flow essentially incompressible} \quad (18)$$

$$* v \gg a : \frac{d\rho}{dx} \text{ large} - \text{flow very compressible}$$

On the other hand, we have for the Parker model [1], from equation (5b),

$$\begin{aligned} r \ll r_* : \frac{1}{v_r} \frac{dv_r}{dr} &\approx 2 \frac{r_*}{r^2} \\ r \gg r_* : \frac{1}{v_r} \frac{dv_r}{dr} &\approx 2 \left(\frac{a^2}{v_r^2} \right) \frac{1}{r}. \end{aligned} \quad (19)$$

Using (19), we have from equation (1) and (11),

$$\frac{1}{\rho} \frac{d\rho}{dr} \approx \begin{cases} -\frac{2}{r} \left(1 + \frac{r_*}{r}\right) \approx -2\frac{r_*}{r^2}, & r \ll r_* \\ -\frac{2}{r} \left(1 + \frac{a^2}{v_r^2}\right) \approx -\frac{2}{r} \left[1 + \frac{1}{4 \log(r/r_*)}\right], & r \gg r_* \end{cases} \quad (20)$$

which implies

$$\begin{aligned} * r \ll r_* : \frac{d\rho}{dr} \text{ large - flow very compressible} \\ * r \gg r_* : \frac{d\rho}{dr} \approx 0 - \text{flow essentially incompressible} \end{aligned} \quad (21)$$

Comparison of (18) and (21) shows that while the flow is very compressible in the sub-critical region and essentially incompressible in the supercritical region in the Parker model [1], the opposite is true for a de Laval nozzle. So, the visualization of an effective de Laval type nozzle associated with the Parker model [1] is valid, barring discrepancies with regard to density variations.

4. Magnetized Stellar Wind

In the MHD model for the stellar wind, the fluid is assumed to have infinite conductivity, zero viscosity and scalar pressure. The star is assumed to have a *monopole* magnetic field that depends only on latitude. The local irregularities in the magnetic field, the polarity reversals and wind velocity fluctuations are ignored. We consider again a steady state model with complete spherical symmetry about the star, and following Weber and Davis [8], the magnetic field is assumed to be distributed in the equatorial plane of the star by the stellar wind and has no component normal to this plane. We focus our attention on the equatorial plane, a flux return at other latitudes notwithstanding. We assume that the magnetic field and flow variables depend only on r , the distance from the star. We assume that there is no energy source/sink present above the base of the corona. We assume that the plasma is fully ionized, the continuum model for the plasma is valid, the perfect gas law holds, and the flow is isothermal. We assume further that the magnetic field and flow variable as well as their derivatives vary continuously so there are no shocks anywhere in the region under consideration.

The equation of conservation of mass (1) implies

$$\rho r^2 v_r = \text{const} = c_1. \quad (22)$$

The Gauss law for the magnetic field gives

$$\nabla \cdot \mathbf{B} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 B_r) = 0 \quad (23a)$$

from which,

$$r^2 B_r = \text{const} = c_2. \quad (23b)$$

Next, the infinite conductivity assumption for the plasma gives

$$\mathbf{E} = -\frac{1}{c}\mathbf{v} \times \mathbf{B} = \langle 0, v_\phi B_r - v_r B_\phi, 0 \rangle. \quad (24)$$

Using (24), Faraday's law

$$\nabla \times \mathbf{E} = \mathbf{0} \quad (25)$$

gives

$$-\frac{1}{r} \frac{\partial}{\partial r} [r (v_\phi B_r - v_r B_\phi)] = 0 \quad (26a)$$

from which,

$$r (v_\phi B_r - v_r B_\phi) = \text{const} = c_3. \quad (26b)$$

Near the coronal base, where the magnetic field is primarily radial, (26b) on using (23b), becomes

$$r (v_\phi B_r - v_r B_\phi) = \Omega_* r^2 B_r = \Omega_* c_2 = c_3 \quad (27a)$$

where Ω_* is the angular velocity of the star into which the magnetic field lines are anchored. (27a) may be rewritten as

$$(v_\phi - \Omega_* r) B_r = v_r B_\phi \quad (27b)$$

which implies that the flow velocity and magnetic field are parallel to each other in the coordinate frame rotating with angular velocity of the star.

Next, the azimuthal component of the equation of momentum gives

$$\rho \left(v_r \frac{dv_\phi}{dr} + \frac{v_r v_\phi}{r} \right) = B_r \frac{dB_\phi}{dr} + \frac{B_r B_\phi}{r} \quad (28a)$$

or

$$\frac{d}{dr} (r v_\phi) = \frac{B_r}{\rho v_r} \frac{d}{dr} (r B_\phi). \quad (28b)$$

Noting from (22) and (23),

$$\frac{B_r}{\rho v_r} = \frac{B_r r^2}{\rho r^2 v_r} = \frac{c_2}{c_1} \quad (29)$$

(28b) becomes

$$r v_\phi - \frac{c_2}{c_1} r B_\phi = \text{const} = L \quad (30)$$

where L is the angular momentum per unit mass carried away by the stellar wind.

Using (27a) and (30), we obtain

$$\left. \begin{aligned} rv_\phi &= \frac{\Omega_* r^2 - (c_2^2/c_2^2 \rho) L}{- (c_1^2/c_2^2 \rho) + 1} \\ rB_\phi &= \frac{\Omega_* r^2 - L}{(c_2/c_1) - (c_1/c_2 \rho)} \end{aligned} \right\}. \quad (31)$$

Noting from (29),

$$\frac{c_1^2}{c_2^2 \rho} = \frac{\rho v_r^2}{B_r^2} \equiv \frac{v_r^2}{v_{A_r}^2} \equiv M_{A_r}^2 \quad (32)$$

(31) becomes

$$\left. \begin{aligned} rv_\phi &= \frac{\Omega_* r^2 - M_{A_r}^2 L}{1 - M_{A_r}^2} \\ rB_\phi &= \frac{\Omega_* r^2 - L}{(c_2/c_1) (1 - M_{A_r}^2)} \end{aligned} \right\}. \quad (33)$$

Noting, at the *Alfvén radius* r_A ,

$$r = r_A : M_{A_r}^2 = 1 \quad (34)$$

the physical compatibility of the solution (33) requires that L is restricted to the value,

$$L = \Omega_* r_A^2. \quad (35)$$

So, r_A may also be interpreted as the lever-arm length of the magnetic braking torque.

Using (35), (33) becomes

$$\left. \begin{aligned} rv_\phi &= \frac{\Omega_* (r^2 - M_{A_r}^2 r_A^2)}{(1 - M_{A_r}^2)} \\ rB_\phi &= \frac{\Omega_* (r^2 - r_A^2)}{(c_2/c_1) (1 - M_{A_r}^2)} \end{aligned} \right\}. \quad (36)$$

For $r \ll r_A$, noting that $M_{A_r}^2 \ll 1$, we obtain from (36),

$$v_\phi \approx \Omega_* r \quad (37)$$

so the stellar wind rotates essentially like a solid body out to the *Alfvén radius*, $r \approx r_A$.

On the other hand, for $r \gg r_A$, noting that $M_{A_r}^2 \gg 1$, we obtain from (36),

$$\begin{aligned}
rv_\phi &= \frac{\Omega_* (r_A^2 - r^2/M_{Ar}^2)}{(1 - 1/M_{Ar}^2)} \\
&\approx \Omega_* (r_A^2 - r^2/M_{Ar}^2) (1 + 1/M_{Ar}^2) \\
&\approx \Omega_* r_A^2 \left[1 - \frac{1}{M_{Ar}^2} \left(\frac{r^2}{r_A^2} - 1 \right) \right].
\end{aligned} \tag{38}$$

Noting from (22) and (23),

$$\frac{M_{Ar}^2}{r^2 v_r} = \left(\frac{\rho v_r^2}{B_r^2} \right) \frac{1}{r^2 v_r} = \frac{\rho v_r r^2}{(B_r r^2)} = \frac{c_1}{c_2^2} = \frac{1}{v_{rA} r_A^2} \tag{39}$$

(38) leads to

$$v_\phi \approx \Omega r_A \left(\frac{r_A}{r} \right) \left(1 - \frac{v_{rA}}{v_r} \right). \tag{40}$$

(40) shows that the coronal plasma does not quite co-rotate with the star out to $r = r_A$ and experience no magnetic torque beyond $r = r_A$. In fact, while much of the angular momentum in the sub-Alfvénic region ($r < r_A$) is carried by the magnetic field, the factor $(1 - v_{rA}/v_r)$ corrects for the angular momentum retained by the magnetic field in the region $r \gg r_A$.

It is of interest to note that the azimuthal velocity profile for the magnetized stellar wind has certain resemblance to that for the hydrodynamic Burgers¹³ vortex (Saffman [17]). From (37) and (40), we have for the azimuthal velocity profile for the magnetized stellar wind,¹⁴

$$v_\phi \approx \begin{cases} \Omega_* r, & r \ll r_A \text{ or } v_r \ll v_{Ar} \\ \frac{\Omega_* r_A^2}{r}, & r \gg r_A \text{ or } v_r \gg v_{Ar}. \end{cases} \tag{41}$$

The azimuthal velocity profile for the hydrodynamic Burgers vortex is

$$v_\phi = \frac{\Gamma}{2\pi r} \left(1 - e^{-ar^2/4\nu} \right) \tag{42}$$

Γ being the circulation associated with the vortex and ν the kinematic viscosity. (42) leads to

$$v_\phi \approx \begin{cases} \left(\frac{\Gamma a}{8\pi\nu} \right) r, & r \ll \sqrt{4\nu/a} \\ \frac{\Gamma}{2\pi r}, & r \gg \sqrt{4\nu/a}. \end{cases} \tag{43}$$

¹³Burgers vortex describes the interplay between the intensification of vorticity due to the imposed straining flow and the diffusion of vorticity due to the action of viscosity.

¹⁴(41) is a more precise result than the one given by Weber and Davis [8] (see the intervals of validity in (41)).

The similarity between (41) and (43) is to be noted. (41) and (43) indeed become identical if one formally identifies Γ , as follows,

$$\Gamma \sim \Omega_* r_\nu^2, \quad r_\nu \sim \sqrt{\nu/a}. \quad (44)$$

Next, for $r \ll r_A$, (36) also gives

$$B_\phi \sim \frac{1}{r} \quad (45)$$

while (23b) gives

$$B_r \sim \frac{1}{r^2}. \quad (46)$$

(45) and (46) imply that, as r increases, the magnetic field will be primarily azimuthal describing the *Parker spirals*.

In the radial component of the equation of momentum, for a weakly-magnetized star like the Sun, the Lorentz force terms are very small compared with the gravitational term and are usually therefore neglected. However, this assumption is not valid for stars with strong magnetic fields. The radial component of the equation of momentum is then

$$\rho \left(v_r \frac{dv_r}{dr} - \frac{v_\phi^2}{r} \right) = -\frac{dp}{dr} - \frac{\rho GM}{r^2} - \frac{B_\phi}{r} \frac{d}{dr} (r B_\phi). \quad (47)$$

Using (33), equation (47) leads to (Belcher and MacGregor [9]),

$$\frac{r}{v_r} \frac{dv_r}{dr} = \frac{(v_r^2 - v_{A_r}^2) (2a^2 + v_\phi^2 - GM/r) + 2v_r v_\phi v_{A_r} v_{A_\phi}}{(v_r^2 - v_{A_r}^2) (v_r^2 - a^2) - v_r^2 v_{A_\phi}^2} \quad (48)$$

where,

$$v_{A_\phi}^2 \equiv \frac{B_\phi^2}{\rho}, \quad v_{A_r}^2 \equiv \frac{B_r^2}{\rho}.$$

Equation (48) may be rewritten as

$$\frac{r}{v_r} \frac{dv_r}{dr} = 2a^2 \frac{(v_r^2 - v_{A_r}^2) \left[r \left\{ 1 + \frac{1}{2a^2} \left[v_\phi^2 + \frac{2v_{A_\phi}^2 \left(\frac{r^2}{r_A^2} \right) \left(1 - \frac{v_r}{v_{A_r}} \right) \right] \right\} - r_* \right]}{r (v_r^2 - c_+^2) (v_r^2 - c_-^2)} \quad (49)$$

where,

$$c_\pm^2 = \frac{v_{A_r}^2 + v_{A_\phi}^2 + a^2}{2} \pm \sqrt{\left(\frac{v_{A_r}^2 + v_{A_\phi}^2 + a^2}{2} \right)^2 - a^2 v_{A_r}^2}. \quad (50)$$

Near the surface of the star, the magnetic field is primarily radial, so $v_{A_\phi} \approx 0$, and (50) becomes

$$c_+ \approx V_{A_r}, \quad c_- \approx a \quad (51)$$

and equation (49) simplifies to,

$$\frac{1}{r} \frac{dv_r}{dr} \approx \frac{2a^2}{r^2} \frac{[r(1 + v_\phi^2/2a^2) - r_*]}{(v_r^2 - a^2)}. \quad (52)$$

For strong rotators ($v_\phi/a \gg 1$), (52) yields at the stellar-wind choking point (where the numerator on the right vanishes),

$$r_c \approx \left(\frac{GM}{\Omega_*^2} \right)^{1/3}. \quad (53)$$

Observe that at this point, the corotation speed $\Omega_* r_c$ of a particle is equal to the circular Keplerian orbit speed $\sqrt{GM/r_c}$ ¹⁵. So, Keplerian orbit conditions prevail near a strong rotator even in a magnetized situation.

On the other hand, for large distances from the star, the magnetic field is mainly azimuthal, so $v_{A_r} \approx 0$, and (50) becomes

$$c_+^2 \approx v_{A_\phi}^2 + a^2, \quad c_- \approx 0 \quad (54)$$

and equation (49) simplifies to ,

$$\frac{1}{v_r} \frac{dv_r}{dr} \approx \frac{2a^2}{r^2} \frac{\left[r \left\{ 1 + \frac{1}{2a^2} \left[v_\phi^2 + \frac{2v_{A_\phi}^2 \left(\frac{r^2}{r_A^2} \right)}{(1 - r^2/r_A^2)(1 + M_{A_r})} \right] \right\} - r_* \right]}{\left[v_r^2 - (v_{A_\phi}^2 + a^2) \right]} \quad (55)$$

Noting,

$$r \gg r_A : M_{A_r}^2 \gg 1 \quad (56)$$

equation (55) simplifies further to,

$$\frac{1}{v_r} \frac{dv_r}{dr} \approx \frac{2a^2}{r^2} \frac{r \left[1 + \frac{1}{2a^2} (v_\phi^2 - 2v_{A_\phi}^2/M_{A_r}) \right] - r_*}{\left[v_r^2 - (v_{A_\phi}^2 + a^2) \right]} \quad (57)$$

Noting,

$$\frac{1}{2a^2} v_\phi^2 \sim \frac{d_1}{r^2}, \quad d_1 > 0 \quad (58a)$$

and assuming,

¹⁵This also corresponds to the balance between outward centrifugal force and the stellar gravity force.

$$\frac{1}{a^2} \frac{v_{A_\phi}^2}{M_{A_r}} \sim \frac{d_2}{r^2}, \quad d_2 > 0 \quad (58b)$$

equation (57) may be rewritten as

$$\frac{1}{v_r} \frac{dv_r}{dr} \approx \frac{2a^2}{r^2} \frac{[r^2 - r_*r + (d_1 - d_2)]}{(v_r^2 - \tilde{a}^2)} \quad (59a)$$

or

$$\frac{1}{v_r} \frac{dv_r}{dr} \approx \frac{2a^2}{r^3} \frac{(r - r_1)(r - r_2)}{(v_r^2 - \tilde{a}^2)} \quad (59b)$$

where,

$$r_{1,2} = \frac{1}{2} \left[r_* \pm \sqrt{r_*^2 - 4(d_1 - d_2)} \right] \gtrless r_* \quad (60)$$

$$\tilde{a}^2 \equiv v_{A_\phi}^2 + a^2.$$

Local Behaviors

(i) Suppose,

$$v_r = \tilde{a} \text{ at } r = r_2 < r_1 \quad (61)$$

Then, for $r \approx r_2$, putting

$$x \equiv r - r_2, \quad y \equiv v_r - \tilde{a} \quad (62)$$

and linearizing in x and y , equation (59) leads to

$$yy' \approx -\frac{a^2}{r_2^3} \alpha x \quad (63)$$

from which,

$$y^2 \approx -\frac{a^2 \alpha}{r_2^3} x^2 \quad (64)$$

where,

$$\alpha \equiv r_1 - r_2 = \sqrt{r_*^2 - 4(d_1 - d_2)}.$$

(64) is unphysical, so (61) cannot happen.

(ii) Suppose, on the other hand,

$$v_r = \tilde{a} \text{ at } r = r_1 > r_2 \text{ and } d_1 > d_2. \quad (65)$$

Then, for $r \approx r_2$, putting

$$x \equiv r - r_2 \quad (66)$$

and linearizing in x , equation (59) leads to

$$\frac{1}{v_r} v_r' \approx -\frac{2a^2}{r_2^3} \frac{\alpha x}{v_r^2 - \tilde{a}^2} \quad (67)$$

which actually shows a wind deceleration in the region $r < r_2$. Equation (67) leads to

$$\frac{v_{r_2}^2 - v_r^2}{\tilde{a}^2} + \ln \left(\frac{v_r^2}{v_{r_2}^2} \right) = \frac{2\alpha}{r_2^3} \left(\frac{a^2}{\tilde{a}^2} \right) x^2. \quad (68)$$

Next, for $r \approx r_1$, putting

$$x \equiv r - r_1, \quad y \equiv v_r - \tilde{a} \quad (69)$$

and linearizing in x and y , equation (59) leads to

$$yy' \approx \frac{a^2}{r_1^3} \alpha x \quad (70)$$

from which,

$$y \approx a \sqrt{\frac{\alpha}{r_1^3}} x \quad (71)$$

which shows the Parker hydrodynamic scenario [1] operational *locally* near $r = r_1$.

On the other hand, in the strong magnetization regime, $d_2 > d_1$, we have from (60), $r_2 < 0$, in which case, the stellar wind deceleration region $r < r_2$ disappears altogether and the Parker hydrodynamic scenario [1] becomes *global* (it now holds, $\forall r$).

5. Discussion

Following Parker's [1] ingenious demonstration that, thanks to a retarding body force (like gravity), a stellar wind can evolve from subsonic to supersonic speeds (as confirmed by *situ* observations from satellites) even in a purely divergent channel (like the interplanetary space) Parker's stellar wind model [1] has been a topic of enormous research.¹⁶ In this paper, we first have revisited Parker's hydrodynamic stellar wind model [1] and made further analytic considerations. We found that the visualization of an effective de Laval type nozzle associated with Parker's model is valid barring discrepancies with regard to density variations.

In view of the strengthening of the magnetization effects away from a star (even when it is weakly-magnetized like the Sun), the interaction of the stellar magnetic field with the expanding corona is of great importance. We have therefore made analytic considerations on the Weber-Davis MHD extension [8] of Parker's model [1] with a view to provide a qualitative understanding of the coupling between the magnetic field and the plasma motion in the stellar wind. We have found that

¹⁶It may be mentioned that there are some observed aspects of the stellar wind that are not fully explained by Parker's model [1]. An example is the fast stellar wind (Feldman et al. [18]), which is channeled out of the coronal holes associated with funnel-like expansion of radial open magnetic field lines (as revealed by SOHO observations, Hassler et al. [19]) and accelerates at a pace considerably faster than that predicted by Parker's model [1].

- * the MHD azimuthal velocity profile actually resembles that for hydrodynamic Burgers vortex;
- * Keplerian-orbit conditions prevail near a strong rotator even in a magnetized situation;
- * Parker's hydrodynamic scenario [1] seems to reappear in the strong magnetization regime.

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